Security of the STARK-friendly hash functions

Anne Canteaut, Tim Beyne, Itai Dinur, Maria Eichlseder, Gregor Leander, Gaetan Leurent, Maria Naya Plasencia, Léo Perrin, Yu Sasaki, Yosuke Todo, et al.

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Motivation

ZK-STARK protocol is expected to be deployed on top of the Ethereum blockchain within the next year

$\longrightarrow$ its security and performance highly depend on the underlying hash function.

Performance. SFH are specified as sequences of low-degree polynomials or low-degree rational maps over a finite field.

Security.

- algebraic attacks based on Gröbner basis [Albrecht et al. 19]...
- all other cryptanalytic techniques.

MPC-friendly, Snark-friendly and Stark-friendly primitives

Objectives:

- minimize the number of multiplications in large fields
- minimize the size of the polynomial relations representing the execution trace over a finite field.

Examples:

- Cradic [Knudsen Nyberg 92], Misty [Matsui 97]
- MiMC [Albrecht et al. 16]

SFH contenders

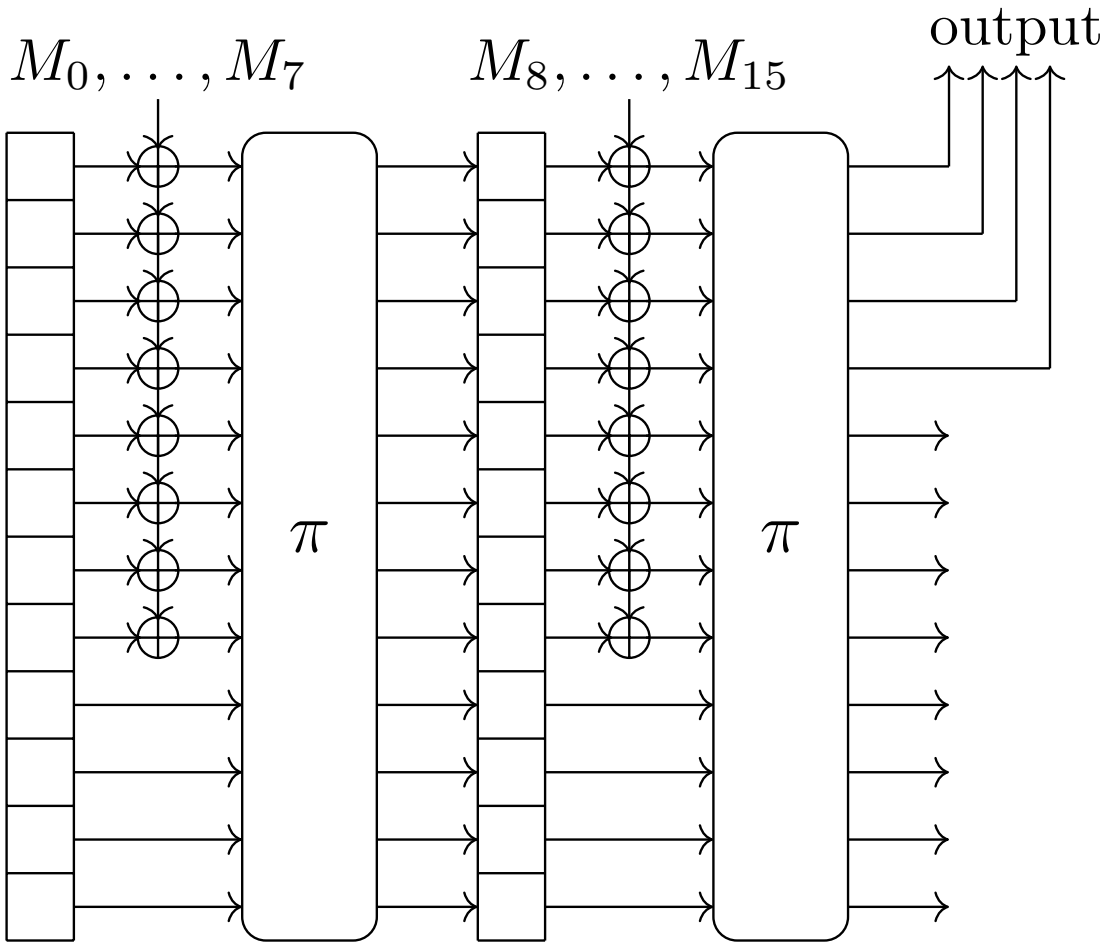
StarkWare challenges `https://starkware.co/hash-challenge/`

Three families of sponges with different permutations

- SPN with large blocks: Vision ($\mathbb{F}_{2^n}$) and Rescue ($\mathbb{F}_p$) [Aly et al. 19]
- HadesMiMC permutation: Starkad ($\mathbb{F}_{2^n}$) and Poseidon ($\mathbb{F}_p$) [Grassi et al. 19]
- GMiMC i.e. $\text{GMiMC}_{\text{erf}}$ over $\mathbb{F}_p$ [Albrecht et al. 19]

Sponge construction

All candidates follow the same sponge construction with blocksize t and capacity c.



Parameters of the sponge

security level	$\log_2 q$	c	t	
128 bits	64	4	12	variant 128-d
	128	2	4	variant 128-a
	128	2	12	variant 128-c
	256	1	3	variant 128-b
	256	1	11	variant 128-e
256 bits	128	4	8	variant 256-a
	128	4	14	variant 256-b

Performance for 128-bit security

Best candidate:

Variant 128-d:

$$t = 12 \text{ and } c = 4 \text{ over } \mathbb{F}_q$$

$$q = \begin{cases} 2^{63} \\ 2^{61} + 20 \times 2^{32} + 1 \end{cases}$$

Compared performance for these parameters

prime fields are more STARK-friendly than binary fields

Prime field:

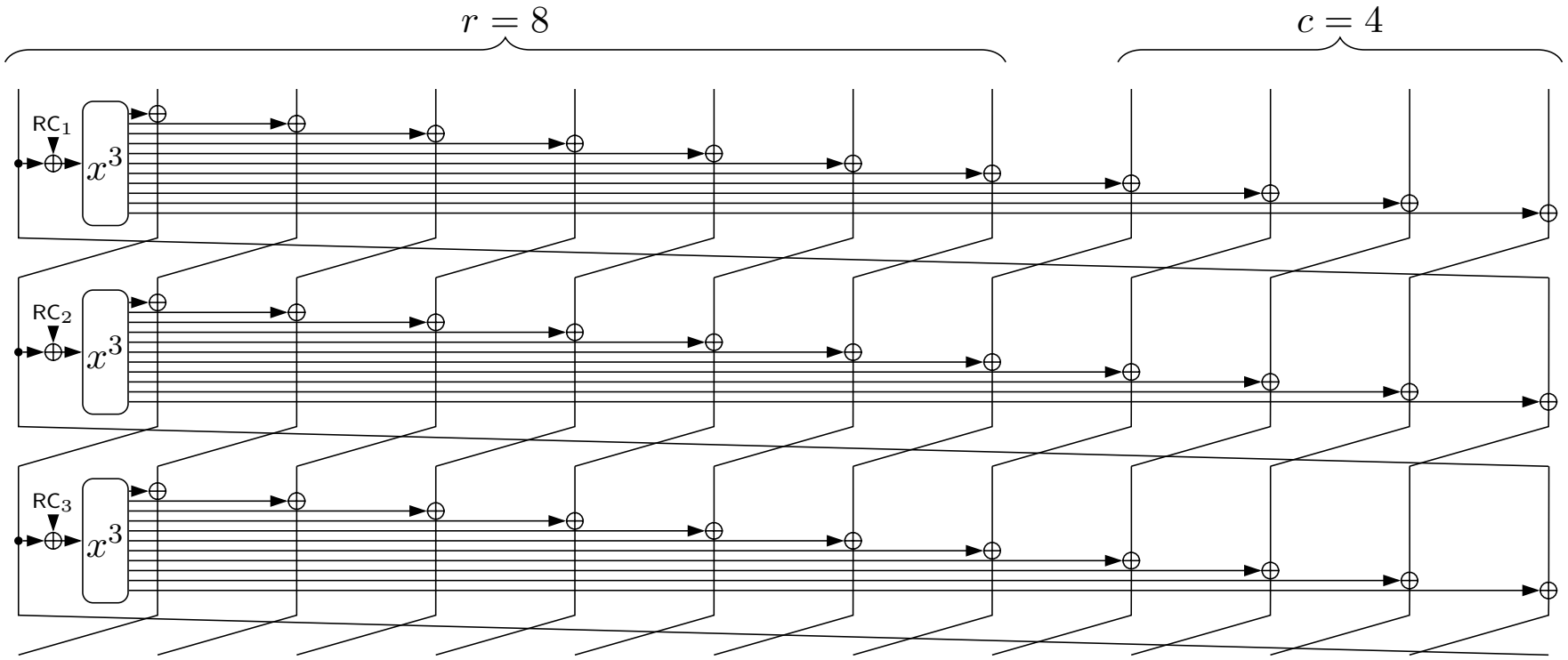
1. GMiMC
2. Rescue
3. Poseidon

Binary field:

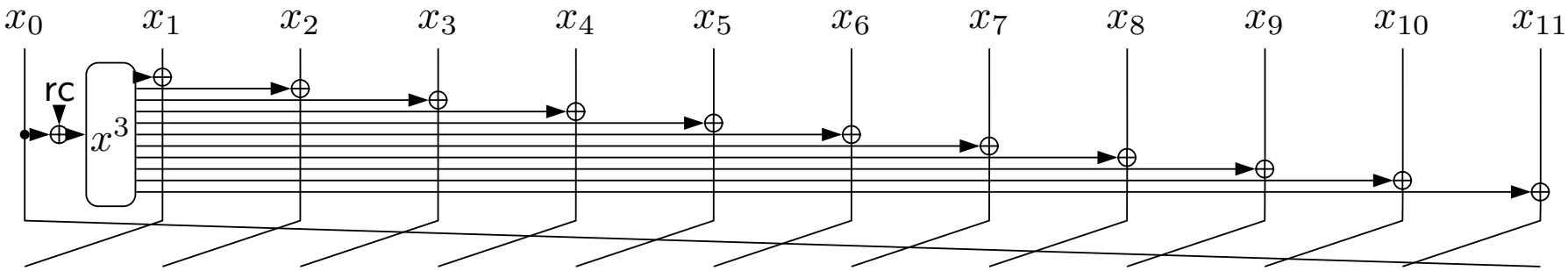
1. Vision
2. Starkad

GMiMC

GMiMC with 101 rounds



A differential distinguisher



Original analysis:

best attack with a characteristic over $(t+1)$ rounds with probability $\left(2q^{-1}\right)^2$.

A better differential:

$$(0\ldots0,\alpha,\alpha')\overset{\mathcal{R}^{t-2}}{\to}(\alpha,\alpha',0\ldots0)\overset{\mathcal{R}}{\to}(\alpha'+\beta,\beta\ldots\beta,\alpha)\overset{\mathcal{R}}{\to}(\beta+\beta'\ldots\beta+\beta',\alpha+\beta',\alpha'+\beta)$$

For $\beta' = -\beta$, we get an iterative differential

$$(0\ldots0,\alpha,\alpha') \xrightarrow{\mathcal{R}^t} (0,\ldots,0,\alpha+\beta,\alpha'+\beta) \text{ with probability } 2q^{-1}$$

A differential distinguisher

With this t-round differential with proba $2q^{-1}$

- A differential characteristic over the 101 rounds with probability 2^{-480} for a 732-bit blocksize.
- With structures, we get valid pairs with complexity 2^{359} (full permutation) and valid pairs with complexity less than 2^{128} for 58 rounds.
- With a rebound-like technique, we expect to get valid pairs conforming with the differential over 58 rounds with complexity close to 2^{64} (on-going work).

Impossible differentials

Original analysis: best impossible differential over $(2t-2)$ rounds

A better impossible differential over $(3t-4)$:

$$(0,\ldots,0,\alpha) \overset{\mathcal{R}^{3t-4}}{\nrightarrow} (\beta,0,\ldots,0)$$

for any nonzero $\alpha \neq \beta$.

Integral attacks over $\mathbb{F}_q$

When $q = 2^n$.

For any (affine) subspace $V \subset \mathbb{F}_2^n$ with $\dim V > \deg F$,

$$\sum_{x \in V} F(x) = 0.$$

Integral attacks over $\mathbb{F}_q$

When $q = 2^n$.

For any (affine) subspace $V \subset \mathbb{F}_2^n$ with $\dim V > \deg F$,

$$\sum_{x \in V} F(x) = 0.$$

Because, for $V = b + \langle a_1, \ldots, a_v \rangle$,

$$D_{a_1} D_{a_2} \ldots D_{a_v} F(b) = \sum_{x \in V} F(x)$$

Not valid in odd characteric.

But for any q

For any exponent k with $0 \leq k \leq q-2$,

$$\sum_{x \in \mathbb{F}_q} x^k = 0$$

General result.

For any $F : \mathbb{F}_q \to \mathbb{F}_q$ with $\deg(F) \leq q-2$,

$$\sum_{x \in \mathbb{F}_q} F(x) = 0 \ .$$

But for any q

For any exponent k with $0 \le k \le q-2$,

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General result.

For any $F : \mathbb{F}_q \to \mathbb{F}_q$ with $\deg(F) \le q-2$,

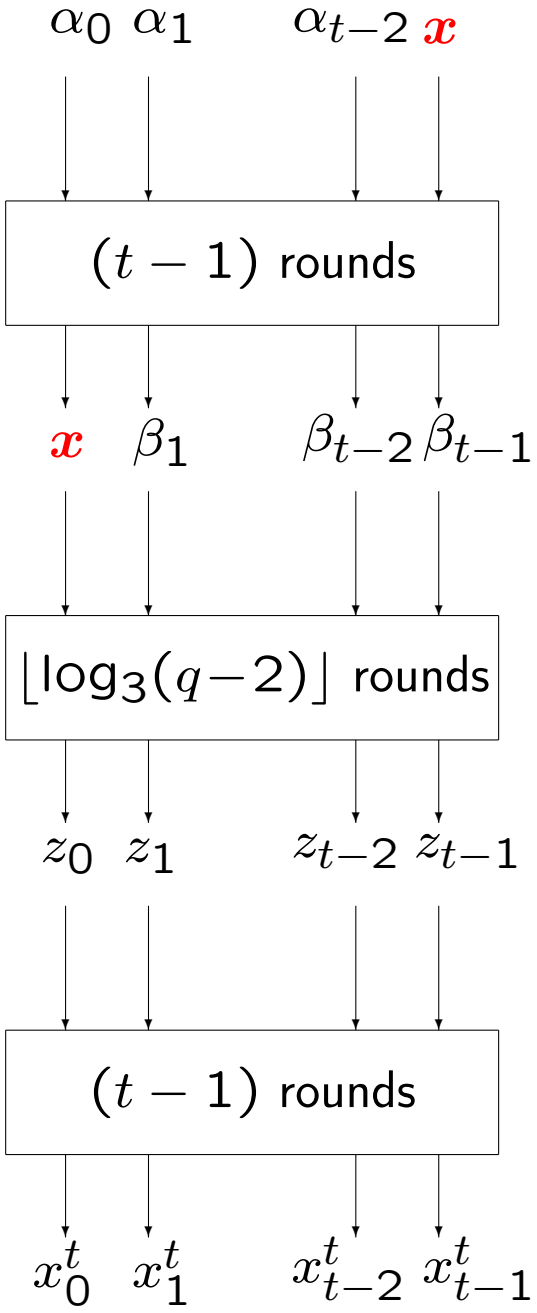
$$\sum_{x \in \mathbb{F}_q} F(x) = 0 \ .$$

Less general than the property over $\mathbb{F}_{2^n}$:

For any (affine) subspace $V \subset \mathbb{F}_2^n$ with $\dim V > \deg F$,

$$\sum_{x \in V} F(x) = 0$$

Integral distinguisher on GMiMC



$x \in \mathbb{F}_q$

polynomial in x
of degree $\leq q-2$

$$Q(x) = \sum_{i=1}^{t-1} x_i^t - (t-2)x_0^t$$

Until the degree does not exceed $(q-2)$

Input set.

$$\mathcal{X} = \{(\alpha_0, \ldots, \alpha_{t-2}, x), x \in \mathbb{F}_q\}$$

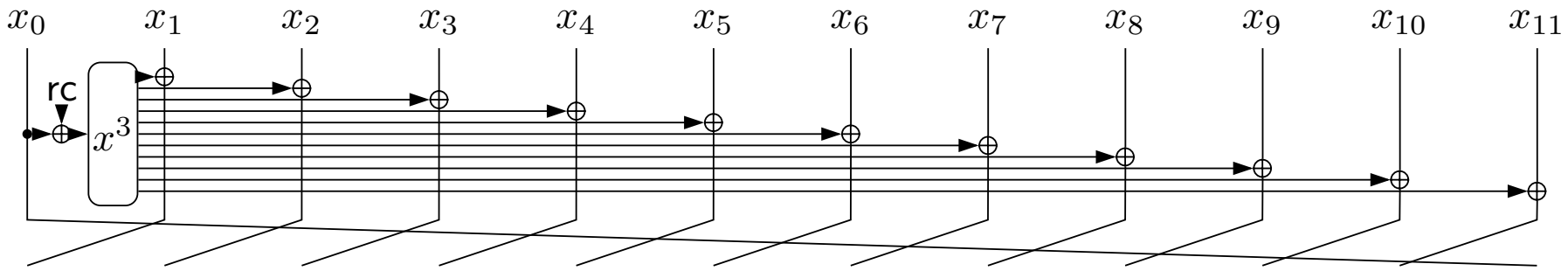
After $(t-1)$ rounds.

$$\mathcal{X}' = \{(x, \beta_1, \ldots, \beta_{t-1}), x \in \mathbb{F}_q\}$$

After r rounds, the degree in x of each branch is at most 3^r.

$\Rightarrow$ all branches are balanced if $3^r \leq q-2$.

Adding $(t-2)$ rounds



The inputs and outputs of Round ℓ satisfy

$$x_i^\ell - x_{i+1}^\ell = x_{i+1}^{\ell-1} - x_{i+2}^{\ell-1}, \quad \forall i \leq t-3$$

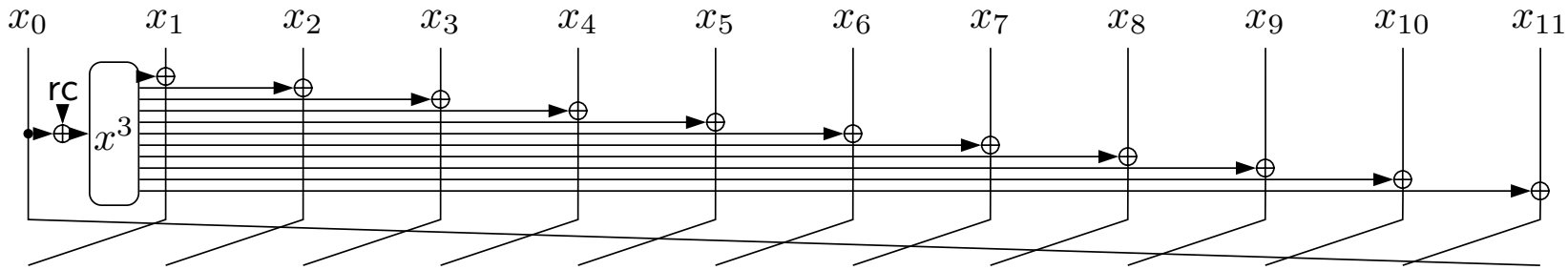
Over $(t-2)$ rounds,

$$x_0^{t-1} - x_1^{t-1} = x_{t-2}^1 - x_{t-1}^1$$

is a polynomial in x of degree $\leq (q-2)$.

$\Rightarrow$ Distinguisher with complexity q on $(2t-3+\lfloor \log_3(q-2) \rfloor)$ rounds (59 rounds)

Adding one more round



The inputs and outputs of Round ℓ satisfy

$$x_i^\ell = x_{i+1}^{\ell-1} + (x_j^\ell - x_{j+1}^{\ell-1}) \text{ and } x_{t-1}^\ell = x_0^{\ell-1}$$

$$\Rightarrow \sum_{i-0}^{t-1} x_i^{\ell-1} - (t-1)x_j^t = \sum_{i=0}^{t-1} x_i^\ell - (t-1)x_{j-1}^\ell$$

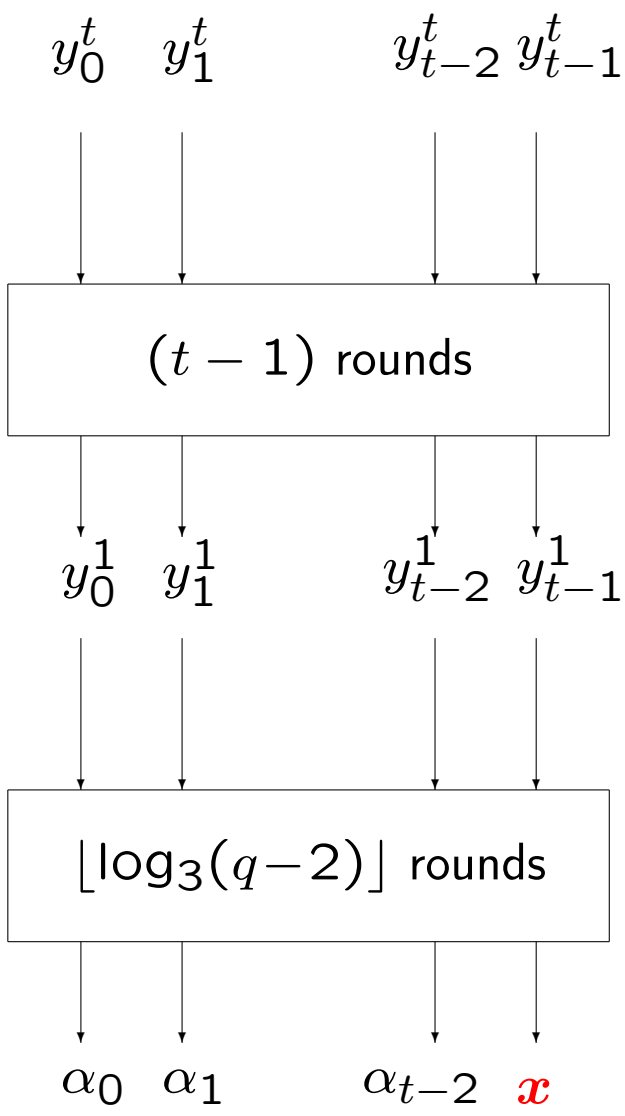
Over $(t-1)$ rounds,

$$\sum_{i=0}^{t-1} x_i^1 - (t-1)x_{t-1}^1 = \sum_{i=0}^{t-1} x_i^t - (t-1)x_0^t$$

$\Rightarrow$ Distinguisher with complexity q on $(2t-2+\lfloor \log_3(q-2) \rfloor)$ rounds (60 rounds)

A few more rounds with two active branches (on-going work).

Computing backwards

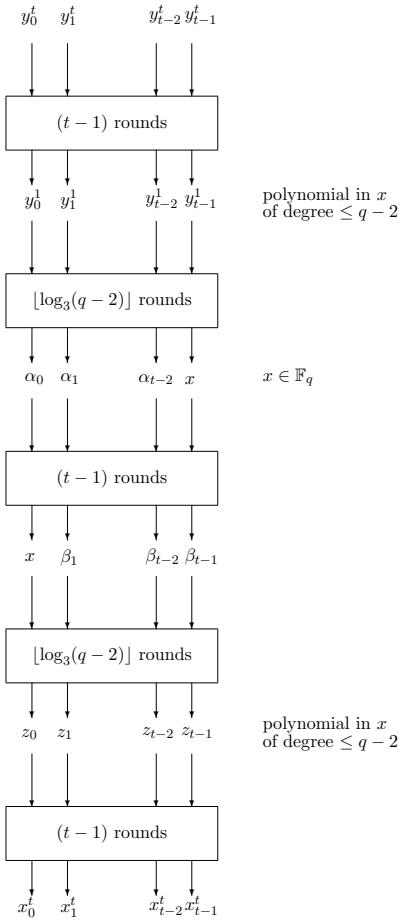


$$Q'(x) = \sum_{i=0}^{t-2} y_i^t - (t-2)y_{t-1}^t$$

$x \in \mathbb{F}_q$

Zero-sum partition on GMiMC on $(3t-3+2\lfloor\log_3(q-2)\rfloor)$ rounds (109)

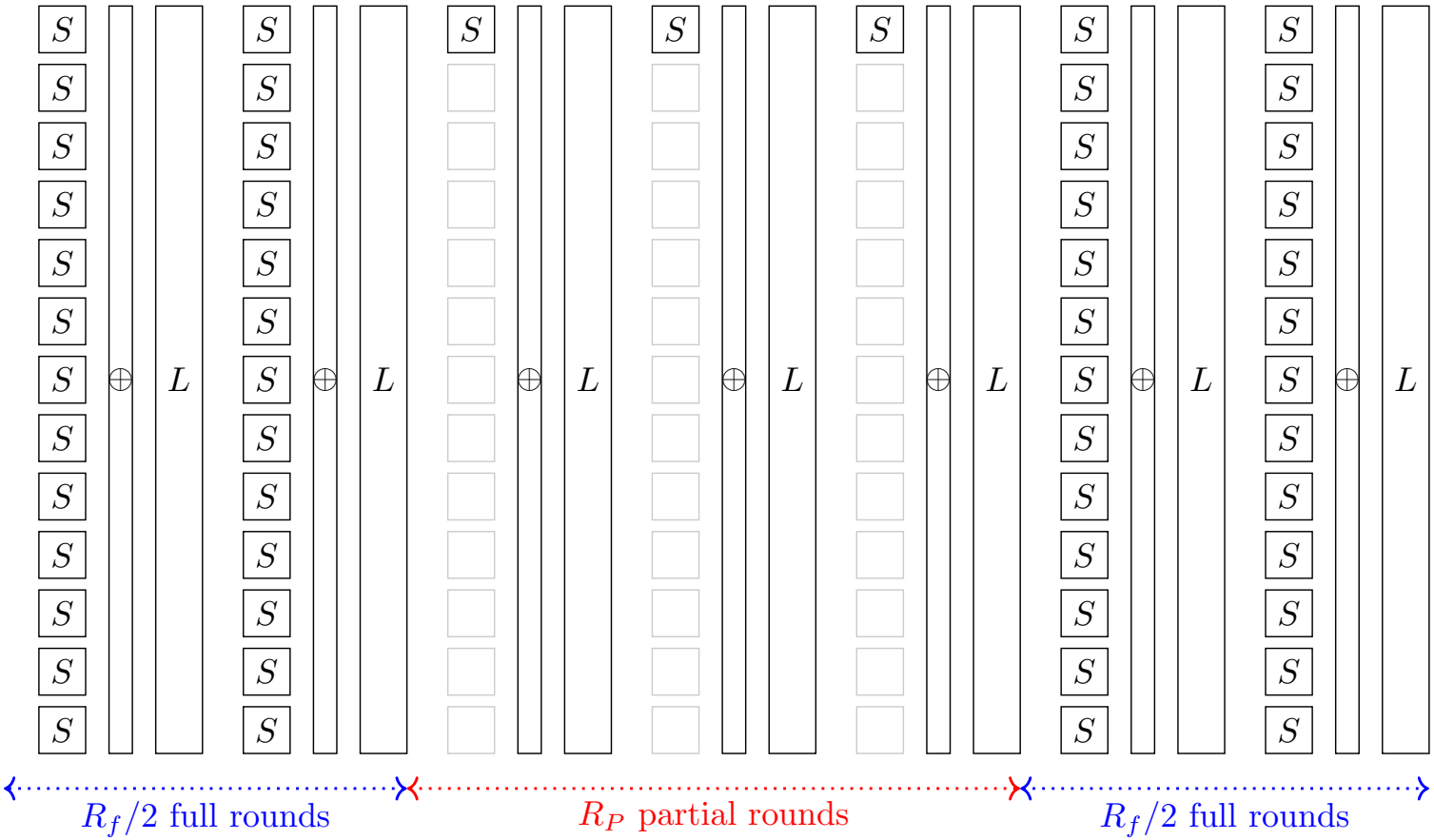
$$\ell(x_0,\ldots,x_{t-1}) = \sum_{i=1}^{t-1} x_i - (t-2)x_0 \text{ sum to } 0$$



$$\ell'(y_0,\ldots,y_{t-1}) = \sum_{i=0}^{t-2} y_i - (t-2)y_{t-1} \text{ sum to } 0$$

HadesMiMC

HadesMiMC



$\boldsymbol{R_f = 8}$ full rounds and $\boldsymbol{R_P = 43}$ (binary) and $\boldsymbol{R_P = 40}$ (prime)

Resistance against statistical attacks

Analysed without the partial rounds.

Differential cryptanalysis:

x^3 has differential uniformity 2 over $\mathbb{F}_q$.

The best differential characteristic satisfies

$$\mathrm{EDP} \leq \left(\frac{2}{q}\right)^{(t+1)R_f/2}$$

$\rightarrow R_f = 6$ are enough.

Degree of the permutation over $\mathbb{F}_q$

Each coordinate is seen as a multivariate polynomial over $\mathbb{F}_q$

After r rounds:

$$\sum_{u=(u_1,\ldots,u_t)} \lambda_u \left(\prod_{i=1}^{t} x_i^{u_i} \right) \text{ where } u_i \leq 3^r$$

$\Rightarrow$ 39 rounds are enough for Poseidon (40 for Starkad) to reach degree $(q-1)$ in each variable

$\Rightarrow \lceil \log_3(t) \rceil$ more rounds are enough to get total degree $(q-1)t$.

Remark: StarkWare challenges with $q \simeq 2^{256}$ and 96 rounds have degree at most 2^{152} in each variable.

Zero-sum partition over $\mathbb{F}_q$

State after the last full Sbox layer before the partial rounds.

$$\mathcal{X} = \{(\alpha_0, \ldots, \alpha_{t-2}, x), x \in \mathbb{F}_q\}$$

After 38 rounds forwards.

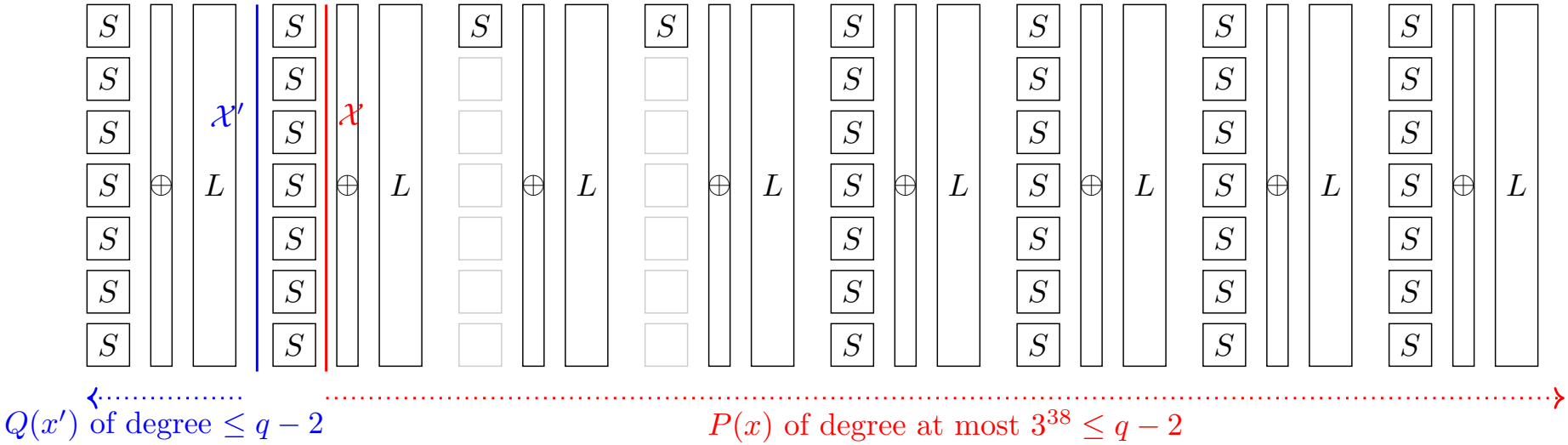
each coordinate has degree at most $(q-2)$.

Computing backwards.

$$S^{-1} : x \mapsto x^s \text{ with } s = \frac{2q-1}{3}$$

$\Rightarrow$ Zero-sum for $R_f = 2 + 4$ and $R_P = 34$ (35 for Starkad).

Zero-sum partition over $\mathbb{F}_q$



Improvement when $q = 2^n$

Each Boolean coordinate is seen as a multivariate polynomial in nt variables over $\mathbb{F}_2$

Degree over $\mathbb{F}_{2^n}$ vs. binary degree.

$$P(x) = \sum_{u \leq 2^n - 1} \lambda_u x^u$$

has binary degree

$$\max\{wt(u) : 0 \leq u < 2^n \text{ and } \lambda_u \neq 0\}$$

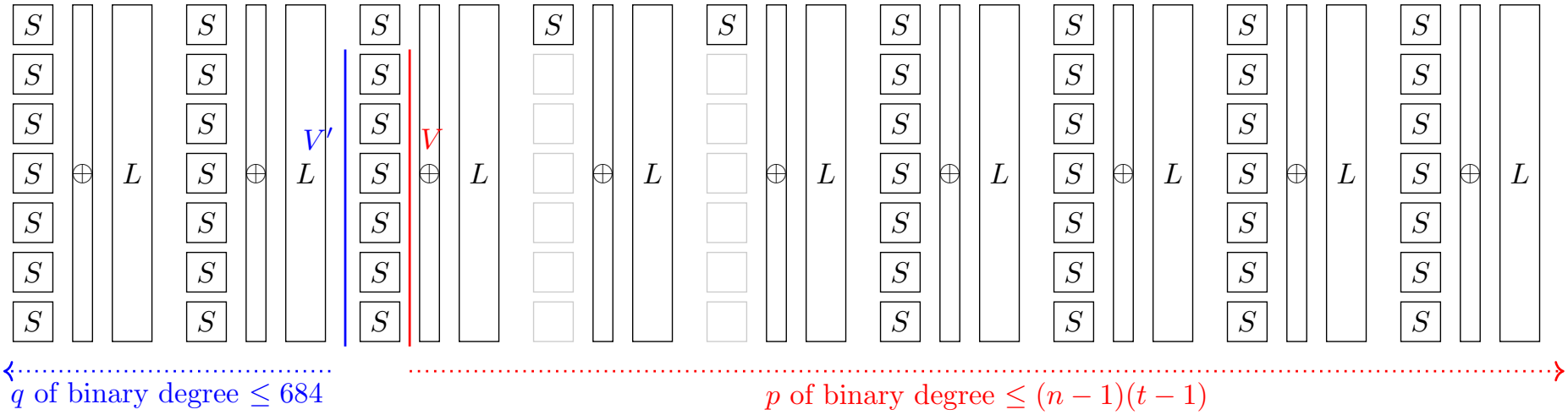
$\Rightarrow$ The inverse Sbox has binary degree $\frac{n+1}{2}$.

Several rounds backwards [Boura, C. 13].

- Two rounds backwards have binary degree ≤ 684
- Three rounds backwards have binary degree ≤ 748

Zero-sum partition over $\mathbb{F}_2$ with $R_f = 3 + 4$ and $R_P = 35$

$$V = \{(0, x_1, \ldots, x_{t-1}), x_i \in \mathbb{F}_{2^n}\} .$$



When the MDS matrix has a small order

How to propagate a subspace through all partial rounds?

Choose V such that all elements in each coset of $L(V)$ have the same value on the first coordinate.

$$L(V) \subset H_0 = \{(0, x_1, \ldots, x_{t-1}), x_i \in \mathbb{F}_q\}$$

or equivalently

$$V \subset \langle M_0 \rangle^{\perp} .$$

We can iterate this R_P times if

$$\mathcal{V} = H_0 \cap \bigcap_{r=0}^{R_P-1} L^r \left(\langle M_0 \rangle^{\perp} \right) \neq \{0\}$$

This holds if $L^r = \mathbf{Id}$ for some $r \leq t-2$.

When the MDS matrix is an involution

The internal states after each partial layer form a coset of V or of $W = L(V)$.

Special choice for V.

$$V = \{(xv_0, \ldots, xv_{t-1}), x \in \mathbb{F}_q\}$$

with $v \in \mathcal{V}$.

$\Rightarrow$ The outputs of the partial rounds vary in a coset of

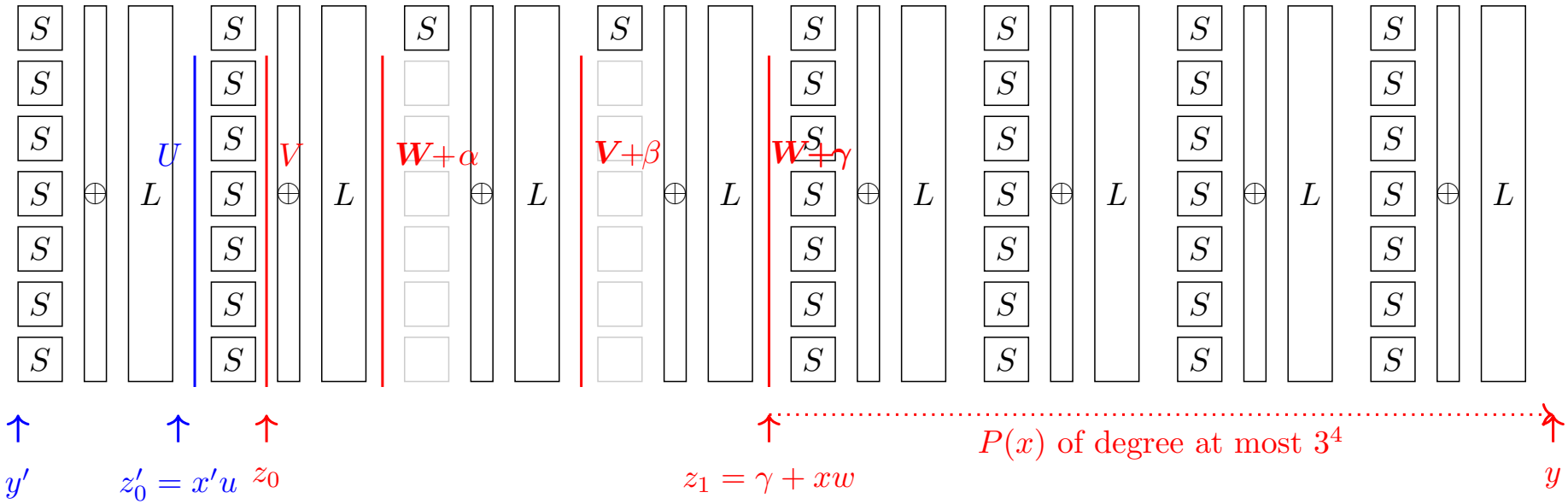
$$\{(xw_0, \ldots, xw_{t-1}), x \in \mathbb{F}_q\}$$

Forward direction.

Each output coordinate is a polynomial in x of degree at most $3^{R_f/2} \leq q - 2$.

$\Rightarrow$ The output coordinates sum to zero.

Zero-sum partition with $R_f = 2 + 4$ and any R_P with complexity q



Open question on the complexity of algebraic attacks

Input: $(a_1, \ldots, a_{t-k}) \in \mathbb{F}_q^{t-k}$ and $(b_1, \ldots, b_k) \in \mathbb{F}_q^k$

Find $x_1, \ldots, x_k \in \mathbb{F}_q^k$ such that

$$\pi(a_1, \ldots, a_{t-k}, x_1, \ldots, x_k) = (b_1, \ldots, b_k, y_1, \ldots y_{t-k}) \text{ for some } y_1, \ldots y_{t-k}$$

Degree of the univariate polynomial of the lexicographical Gröbner basis [Faugère-Perret].

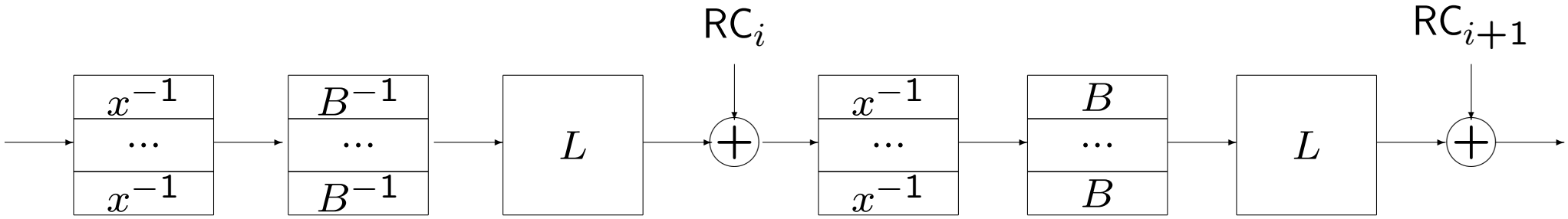
$$D = 3^{kR_f + R_P - 2k + 1}$$

Complexity for solving the system $= D^2$.

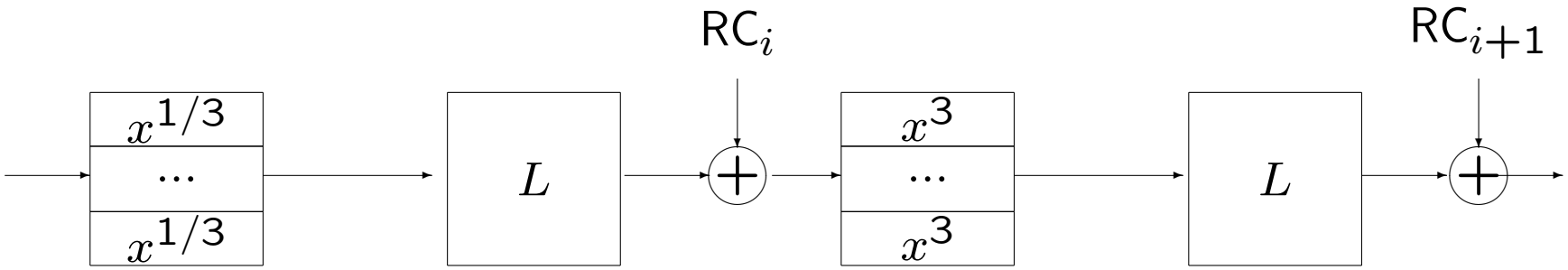
Variants aiming at 256-bit security have $D \simeq 2^{170}$.

Vision and Rescue

Vision (20 rounds)



Rescue (20 rounds)



Degree of Rescue

Activate one input coordinate $x \in \mathbb{F}_p$

After one round.

$$\lambda x^{1/3} + \mu$$

$\Rightarrow$ degree $\frac{2p-1}{3}$

After the second Sbox layer.

$$(\lambda x^{1/3} + \mu)^3$$

which contains only monomials $x^{1/3}$, $x^{2/3}$, x and a constant term.

$x^{2/3}$ has degree $\frac{p+1}{3}$.

$\Rightarrow$ The degree does not increase between the first and second round.

But even by activiting more inputs, we cannot find an integral attack on more than 4 rounds.

Conclusions

We need to find the right tools for analyzing symmetric primitives over non-binary fields:

- linear attacks and their variants?
- more general integral attacks?

Open question:

Does the form of q affect the security?

For instance, if $p = 2^{2^n} + 1$?